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TRANSVERSALS IN UNIFORM HYPERGRAPHS WITH PROPERTY $(p,2)$

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Dedicated to the memory of Paul Erdős

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Let $f(r,p,t)$ $(p>t\geq 1,\ r\geq 2)$ be the maximum of the cardinality of a minimum transversal over all r-uniform hypergraphs $\mathcal{H}$ possessing the property that every subhypergraph of $\mathcal{H}$ with p edges has a transversal of size t. The values of $f(r,p,2)$ for $p=3,4,5,6$ were found in [1] and bounds on $f(r,7,2)$ are given in [3]. Here we prove that $f(r,p,2)\leq 1.3\frac{r}{p^{0.5}-o(p^{0.5})}$ for large p and huge r.

1. Introduction

A *transversal* of a family $\mathcal{F}$ of sets is a subset of $\bigcup_{F\in\mathcal{F}} F$ meeting all members of $\mathcal{F}$. The smallest cardinality $\tau(\mathcal{F})$ of a transversal of $\mathcal{F}$ is called the *transversal number* of $\mathcal{F}$. For a hypergraph $\mathcal{H}=(V,\mathcal{E})$, a *transversal* is a transversal of $\mathcal{E}$.

Say that a family $\mathcal{B}$ *possesses the property* (p,t) if $\tau(\mathcal{F})\leq t$ for every $\mathcal{F}\subset\mathcal{B}$ with $|\mathcal{F}|=p$. Property $(p,1)$ means that any p members of $\mathcal{B}$ have a common element. Erdős, Hajnal, and Tuza [2] raised the following problem:

For given integers r, p, and t $(p>t\geq 1,\ r\geq 2)$, determine the largest value, $f(r,p,t)$, of $\tau(\mathcal{F})$ taken over the class of r-uniform families $\mathcal{F}$ possessing the property (p,t).

It is known that $f(r,p,1)=\lceil r/(p-1)\rceil$ (see, e.g. [5, Ex.13.25(b)]). For $t>1$ the picture is less clear. Erdős, Fon-Der-Flaass, Kostochka, and Tuza [1]

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determined the exact values of $f(r,p,2)$ for $3 \le p \le 6$ (in particular, $f(r,6,2) = r$) and proved the following general bounds.

Theorem A [1]. *(i) For every r,p and t, $f(r,p,t) > rp^{-1/t}$;*
(ii) for $r > p > 2^t$, $f(r,p,t) \le t\left\lceil r(\lfloor p^{1/t}\rfloor - 1)^{-1}\right\rceil$.

It appeared that for $p = 3,4,5,6$, the 'worst' hypergraphs with property $(p,2)$ are complete r-uniform hypergraphs, i.e. if $h(r,p,t)$ is the maximum n such that the complete r-uniform hypergraph K_n^r possesses property $(p,2)$, then $f(r,p,2) = \tau(K^r_{h(r,p,2)}) = h(r,p,2) - r + 1$. The lower bound in Theorem A also was obtained from bounds on $h(r,p,t)$. This suggested that $f(r,p,2) = \tau(K^r_{h(r,p,2)}) = h(r,p,2) - r + 1$ for every p and large r, but this is not the case. It was proved in [3], that for $k \ge 10$, $f(4k,7,2) \ge 3k+1$ while the worst complete $4k$-uniform hypergraph possessing property $(7,2)$ has transversal number $3k$. It was also shown in [3] that $f(r,7,2) \le \lceil \frac{7r}{8} \rceil$.

In spite of the examples for $f(r,7,2)$, Erdős thought that probably for large r and fixed p and t, complete r-uniform hypergraphs are 'close' to the 'worst' ones. The first problem in [1] was stated as follows:

Prove or disprove that $h(r,p,t) \ge (1-o(1))f(r,p,t) + r$ when r tends to infinity while p and t are fixed.

The aim of this paper is to supply evidence in favor of Erdős' insight for $t = 2$. We prove

Theorem 1. *If $p < 0.01r^{2/3}$, then*

$$f(r,p,2) \le \frac{3\sqrt{3}}{4}\frac{r}{\sqrt{p} - o(\sqrt{p})} \le 1.3\frac{r}{\sqrt{p} - o(\sqrt{p})}.$$

Recall that Theorem A gives $\frac{r}{\sqrt{p}} < f(r,p,2) \le \frac{2r}{\sqrt{p} - o(\sqrt{p})}$ and the lower bound is given by $h(r,p,2)$.

The structure of the paper is as follows. In the next section we prove (for completeness) a folklore lemma on covering edges of complete graphs, restate Theorem 1 in a form more convenient for the proof and start the proof. In Section 3, we derive useful properties of 'large' subfamilies of a hypothetical counterexample to the theorem, and in Section 4 finish the proof of the main result.

2. Preliminaries

The following result by Iwaniec and Pintz [4] will be helpful.

Lemma 1. [4] *There exists n_0 such that for every real $x \geq n_0$, the interval $[x-x^{23/42}, x]$ contains a prime number.*

Lemma 2. *Let $2r^{2/3} < f < r$. Then to cover all the edges of K_r by f-vertex subsets, $(r/f)^2 + o((r/f)^2)$ of those subsets suffice.*

Proof. Let $x = r/f$ be sufficiently large. Then by Lemma 1, there exists a prime $q \in [x+1, x+2x^{23/42}]$. Partition $V(K_r)$ into q^2+q+1 subsets $V_1, \ldots, V_{q^2+q+1}$ of cardinalities $\lfloor \frac{r}{q^2+q+1} \rfloor$ and $\lceil \frac{r}{q^2+q+1} \rceil$. Let F_q be a projective plane of order q, whose lines are $L_1, \ldots, L_{q^2+q+1}$ and points are $v_1, \ldots, v_{q^2+q+1}$. For $i = 1, \ldots, q^2+q+1$, let L'_i be obtained from L_i by replacing every $v_j \in L_i$ by the set V_j. Then, by the construction, the sets $L'_1, \ldots, L'_{q^2+q+1}$ cover all the edges of K_r. Since $q \leq x + 2x^{23/42} \leq r^{1/3}$, for each i,

$$|L'_i| \leq (q+1)\left\lceil \frac{r}{q^2+q+1} \right\rceil < \frac{r(q+1)}{q^2+q+1} + q + 1 =$$
$$= \frac{r}{q-1} - \frac{r(q+2)}{(q-1)(q^2+q+1)} + q + 1 \leq \frac{r}{x} + \left(q + 1 - \frac{r(q+2)}{q^3-1}\right) \leq f + 0.$$

Note that the number of sets is $q^2+q+1 \leq (x+2x^{23/42})^2 + x + 2x^{23/42} + 1 = x^2 + o(x^2)$. This proves the lemma. ∎

The main result of the paper will be easier to prove in the following form.

Theorem 2. *Let $10r^{2/3} < f < r$ and $\mathcal{B}$ be a family of r-element sets. If $\tau(\mathcal{B}) > f$, then*

$$(1) \qquad \exists\, \mathcal{F} \subset \mathcal{B} \text{ such that } \tau(\mathcal{F}) > 2 \text{ and } |\mathcal{F}| \leq \frac{27}{16}\frac{r^2}{f^2} + o\left(\frac{r^2}{f^2}\right).$$

Since we are interested only in large f and r/f, we will prove Theorem 2 for r and f such that $\frac{f}{6}$ and $\frac{r}{48f}$ are integers. That would save us some floors and ceilings.

So, let $\mathcal{B}$ be a family of r-element sets with $\tau(\mathcal{B}) > f$. In the next section we show that (1) holds if $\mathcal{B}$ contains some subfamilies of special kinds.

3. Subfamilies of $\mathcal{B}$ with large transversal number

For each family $\mathcal{D}$ and set A, we denote

$$\mathcal{D}_A = \{B \in \mathcal{D} \mid B \cap A = \emptyset\}.$$

Lemma 3. *If there exists* $\mathcal{B}' \subset \mathcal{B}$ *with* $\tau(\mathcal{B}') > f/3$ *such that* $|A_1 \cap A_2| \geq \frac{61r}{72}$ *for every* $A_1, A_2 \in \mathcal{B}'$, *then (1) holds.*

Proof of Lemma 3. Assume that such $\mathcal{B}' \subset \mathcal{B}$ exists. Fix $A \in \mathcal{B}'$. Let $\{A_1, \ldots, A_s\}$ be a minimum family of subsets of size f covering all pairs of elements of A. By Lemma 2, $s = (r/f)^2 + o((r/f)^2)$. Since $\tau(\mathcal{B}) > f$, for each $i = 1, \ldots, s$, there exists $H_i \in \mathcal{B}_{A_i}$.

Partition A into $3r/f$ parts $A(j)$, $j = 1, \ldots, 3r/f$ of size $f/3$. Since $\tau(\mathcal{B}') > f/3$, for each $j = 1, \ldots, 3r/f$, there exists $B_j \in \mathcal{B}'_{A(j)}$. For every j, since under conditions of the lemma, $|B_j \setminus A| \leq \frac{11r}{72}$, we can partition $B_j \setminus A$ into $\frac{11r}{48f}$ parts $A_{i,j}$, $i = 1, \ldots, \frac{11r}{48f}$ of size at most $2f/3$. Since, for each i, $|A(j) \cup A_{i,j}| \leq f$, there exists $C_{i,j} \in \mathcal{B}_{A(j) \cup A_{i,j}}$.

Define $\mathcal{F} = \{A\} \cup \{B_j \mid j = 1, \ldots, 3r/f\} \cup \{C_{i,j} \mid i = 1, \ldots, \frac{11r}{48f}; j = 1, \ldots, 3r/f\} \cup \{H_i \mid i = 1, \ldots, s\}$. Then

$$|\mathcal{F}| \leq 1 + \frac{3r}{f} + \frac{11r^2}{16f^2} + \left(\frac{r}{f}\right)^2 + o\left(\left(\frac{r}{f}\right)^2\right) = \frac{27r^2}{16f^2} + o\left(\frac{r^2}{f^2}\right).$$

Suppose that there are two elements α and β meeting all the members of $\mathcal{F}$. At least one of them (say α) belongs to A. If $\beta \in A$, too, then one of H_i is not covered. So, $\beta \notin A$. By construction, there exists $A(j), 1 \leq j \leq 3r/f$, such that $\alpha \in A(j)$. In order to cover B_j, it is necessary that $\beta \in B_j \setminus A$. Let $\beta \in A_{i,j}$. Then $\{\alpha, \beta\}$ does not meet $C_{i,j}$, a contradiction. This proves the lemma. ∎

Lemma 4. *If there exists* $\mathcal{B}' \subset \mathcal{B}$ *with* $\tau(\mathcal{B}') > f/3$ *such that*

$$|A_1 \cap A_2 \cap A_3| \geq \frac{31}{40} r \quad \text{for every} \quad A_1, A_2, A_3 \in \mathcal{B}', \tag{2}$$

then (1) holds.

Proof of Lemma 4. Assume that such $\mathcal{B}' \subset \mathcal{B}$ exists. Let $D_1, D_2 \in \mathcal{B}'$ be such that $|D_1 \cap D_2| = \min\{|A_1 \cap A_2| : A_1, A_2 \in \mathcal{B}'\}$.

Let $x = \frac{1}{r}|D_1 \cap D_2|$. If $x \geq \frac{61}{72}$, then by Lemma 3 we are done. Let

$$\frac{31}{40} \leq x \leq \frac{61}{72}.$$

Let $\{A_1, \ldots, A_s\}$ be a minimum family of subsets of size f covering all pairs of elements in $D_1 \cap D_2$. By Lemma 2, $s = (xr/f)^2 + o((r/f)^2)$. Since $\tau(\mathcal{B}) > f$, for each $i = 1, \ldots, s$, there exists $H_i \in \mathcal{B}_{A_i}$.

Partition $D_1 \cap D_2$ into $\left\lceil \frac{3xr}{f} \right\rceil$ parts $A(j)$, $j=1,\dots,\left\lceil \frac{3xr}{f} \right\rceil$ of size at most $f/3$. Since $\tau(\mathcal{B}')>f/3$, for each $j=1,\dots,\left\lceil \frac{3xr}{f} \right\rceil$, there exists $B_j \in \mathcal{B}'_{A(j)}$. By (2), for every j,

$$|B_j \setminus (D_1 \cap D_2)| \le \frac{9r}{40}.$$

Partition every $B_j \setminus (D_1 \cap D_2)$ into at most $\left\lceil \frac{27r}{80f} \right\rceil$ parts $A_{k,j}$, $k=1,\dots,\left\lceil \frac{27r}{80f} \right\rceil$ of size at most $2f/3$. Since, for each k, $|A(j) \cup A_{k,j}| \le f$, there exists $C_{k,j} \in \mathcal{B}_{A(j) \cup A_{k,j}}$.

Now for $i=1,2$, partition $D_i \setminus D_{3-i}$ into $\left\lceil \frac{2(1-x)r}{f} \right\rceil$ parts $A(j,i)$, $j=1,\dots,\left\lceil \frac{2(1-x)r}{f} \right\rceil$ of size $f/2$. Since $\tau(\mathcal{B})>f$, for each j_1,j_2, there exists $B(j_1,j_2) \in \mathcal{B}_{A(j_1,1)\cup A(j_2,2)}$. Let

$$\begin{aligned}\mathcal{F} = \{D_1,D_2\} \cup \left\{B_j \mid j=1,\dots,\left\lceil \frac{3xr}{f} \right\rceil\right\}\\ \cup \left\{C_{k,j} \mid k=1,\dots,\left\lceil \frac{27r}{80f} \right\rceil; j=1,\dots,\left\lceil \frac{3xr}{f} \right\rceil\right\}\\ \cup \left\{B(j_1,j_2) \mid j_1,j_2=1,\dots,\left\lceil \frac{2(1-x)r}{f} \right\rceil\right\} \cup \{H_i \mid i=1,\dots,s\}.\end{aligned}$$

Then

$$\begin{aligned}|\mathcal{F}| \le 2 + \left(1+\frac{3rx}{f}\right) + \left(1+\frac{3rx}{f}\right)\left(1+\frac{27r}{80f}\right) + \left(1+\frac{2(1-x)r}{f}\right)^2 +\\ + \left(\frac{xr}{f}\right)^2 + o\left(\left(\frac{r}{f}\right)^2\right) = \left(\frac{81x}{80} + 4(1-x)^2 + x^2\right)\frac{r^2}{f^2} + o\left(\frac{r^2}{f^2}\right).\end{aligned}$$

Let $g(x) = \frac{81x}{80} + 4(1-x)^2 + x^2$. Since $\frac{31}{40} \le x \le \frac{61}{72}$, we have $g(x) \le \max\left\{g\left(\frac{31}{40}\right), g\left(\frac{61}{72}\right)\right\}$. Now,

$$\begin{aligned}g\left(\frac{61}{72}\right) &= \frac{81\cdot 61}{80\cdot 72} + 4\left(\frac{11}{72}\right)^2 + \left(\frac{61}{72}\right)^2 \le 0.858+0.094+0.718 = 1.67 < \frac{27}{16},\\ g\left(\frac{31}{40}\right) &= \frac{81\cdot 31}{80\cdot 40} + 4\left(\frac{9}{40}\right)^2 + \left(\frac{31}{40}\right)^2 = \frac{2511+648+1922}{3200} < 1.6 < \frac{27}{16}.\end{aligned}$$

Suppose that there are two elements α and β meeting all the members of $\mathcal{F}$. At least one of them (say α) belongs to D_1. Suppose first that $\alpha \in D_1 \cap D_2$, say $\alpha \in A(j)$. If $\beta \in D_1 \cap D_2$, too, then one of H_i is not covered. So, $\beta \notin D_1 \cap D_2$. In order to cover B_j, it is necessary that $\beta \in B_j \setminus (D_1 \cap D_2)$. Let $\beta \in A_{k,j}$. Then $\{\alpha,\beta\}$ does not meet $C_{k,j}$, a contradiction.

Thus, we may assume that $\alpha\in D_1\setminus D_2$ and that $\beta\notin D_1\cap D_2$ (otherwise we swap the roles of α and β). Then $\beta\in D_2\setminus D_1$. If $\alpha\in A(j_1,1)$ and $\beta\in A(j_2,2)$, then $\{\alpha,\beta\}$ does not meet $B(j_1,j_2)$. This proves the lemma. ∎

The following lemma has a similar proof.

Lemma 5. *If there exists $\mathcal{B}'\subset\mathcal{B}$ with $\tau(\mathcal{B}')>f/3$ such that*

$$|A_1\cap A_2\cap A_3\cap A_4|\geq\frac{3r}{4}\quad\text{for every}\quad A_1,A_2,A_3,A_4\in\mathcal{B}',\tag{3}$$

then (1) holds.

Proof of Lemma 5. Assume that such $\mathcal{B}'\subset\mathcal{B}$ exists. Let $D_1,D_2,D_3\in\mathcal{B}'$ be such that $|D_1\cap D_2\cap D_3|=\min\{|A_1\cap A_2\cap A_3|:A_1,A_2,A_3\in\mathcal{B}'\}$.

Let $y=\frac{1}{r}|D_1\cap D_2\cap D_3|$. If $y\geq\frac{31}{40}$, then by Lemma 4 we are done. Let

$$\frac{3}{4}\leq y\leq\frac{31}{40}.$$

Let $\{A_1,\dots,A_s\}$ be a minimum family of subsets of size f covering all pairs of elements in $D_1\cap D_2\cap D_3$. By Lemma 2, $s=(yr/f)^2+o((r/f)^2)$. Since $\tau(\mathcal{B})>f$, for each $i=1,\dots,s$, there exists $H_i\in\mathcal{B}_{A_i}$.

Partition $D_1\cap D_2\cap D_3$ into $\left\lceil\frac{3yr}{f}\right\rceil$ parts $A(j)$, $j=1,\dots,\left\lceil\frac{3yr}{f}\right\rceil$ of size at most $f/3$. Since $\tau(\mathcal{B}')>f/3$, for each $j=1,\dots,\left\lceil\frac{3yr}{f}\right\rceil$, there exists $B_j\in\mathcal{B}'_{A(j)}$. By (3), for every j,

$$|B_j\setminus(D_1\cap D_2\cap D_3)|\leq\frac{r}{4}.$$

Partition every $B_j\setminus(D_1\cap D_2\cap D_3)$ into at most $\left\lceil\frac{3r}{8f}\right\rceil$ parts $A_{k,j}$, $k=1,\dots,\left\lceil\frac{3r}{8f}\right\rceil$ of size at most $2f/3$. Since, for each k, $|A(j)\cup A_{k,j}|\leq f$, there exists $C_{k,j}\in\mathcal{B}_{A(j)\cup A_{k,j}}$.

Let $z=\frac{1}{r}(|D_1\cap D_2|-|D_1\cap D_2\cap D_3|)$. For $i=1,2$, partition $D_i\setminus D_{3-i}$ into $\left\lceil\frac{2(1-y-z)r}{f}\right\rceil$ parts $A(j,i)$, $j=1,\dots,\left\lceil\frac{2(1-y-z)r}{f}\right\rceil$ of size at most $f/2$. Since $\tau(\mathcal{B})>f$, for each j_1,j_2, there exists $B(j_1,j_2)\in\mathcal{B}_{A(j_1,1)\cup A(j_2,2)}$.

Finally, partition $D_1\cap D_2$ into $\left\lceil\frac{2zr}{f}\right\rceil$ parts $M(j,1,2)$, $j=1,\dots,\left\lceil\frac{2zr}{f}\right\rceil$ of size at most $f/2$ and partition $D_3\setminus(D_1\cap D_2\cap D_3)$ into $\left\lceil\frac{2(1-y)r}{f}\right\rceil$ parts $M(j,3)$, $j=1,\dots,\left\lceil\frac{2(1-y)r}{f}\right\rceil$ also of size at most $f/2$. Since $\tau(\mathcal{B})>f$, for each j_1,j_2, there exists $L(j_1,j_2)\in\mathcal{B}_{M(j_1,1.2)\cup M(j_2,3)}$.

Let

$$\begin{aligned}\mathcal{F} = \{D_1, D_2, D_3\} &\cup \left\{B_j \mid j = 1, \ldots, \left\lceil \frac{3yr}{f} \right\rceil \right\}\\ &\cup \left\{C_{k,j} \mid k = 1, \ldots, \left\lceil \frac{3r}{8f} \right\rceil ; j = 1, \ldots, \left\lceil \frac{3yr}{f} \right\rceil \right\}\\ &\cup \left\{B(j_1, j_2) \mid j_1, j_2 = 1, \ldots, \left\lceil \frac{2(1-y-z)r}{f} \right\rceil \right\}\\ &\cup \left\{L(j_1, j_2) \mid j_1 = 1, \ldots, \left\lceil \frac{2zr}{f} \right\rceil ; j_2 = 1, \ldots, \left\lceil \frac{2(1-y)r}{f} \right\rceil \right\}\\ &\cup \{H_i \mid i = 1, \ldots, s\}.\end{aligned}$$

Then

$$\begin{aligned}|\mathcal{F}| \le 3 &+ \left(1 + \frac{3ry}{f}\right) + \left(1 + \frac{3ry}{f}\right)\left(1 + \frac{3r}{8f}\right) + \left(1 + \frac{2(1-y-z)r}{f}\right)^2 +\\ &\left(1 + \frac{2zr}{f}\right)\left(1 + \frac{2(1-y)r}{f}\right) + \left(\frac{yr}{f}\right)^2 + o\left(\left(\frac{r}{f}\right)^2\right)\\ &\le \left(\frac{9y}{8} + 4(1-y)^2 + y^2\right)\frac{r^2}{f^2} + o\left(\frac{r^2}{f^2}\right).\end{aligned}$$

Let $g(y) = \frac{9y}{8} + 4(1-y)^2 + y^2$. Since $\frac{3}{4} \le y \le \frac{31}{40}$, we have $g(y) \le \max\left\{g\left(\frac{3}{4}\right), g\left(\frac{31}{40}\right)\right\}$. Now,

$$g\left(\frac{3}{4}\right) = \frac{27}{32} + 4\left(\frac{1}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{27}{32} + \frac{4}{16} + \frac{9}{16} = \frac{26.5}{16},$$

$$g\left(\frac{31}{40}\right) = \frac{9 \cdot 31}{8 \cdot 40} + 4\left(\frac{9}{40}\right)^2 + \left(\frac{31}{40}\right)^2 = \frac{1395 + 324 + 961}{1600} = 1.675 < \frac{27}{16}.$$

Suppose that there are two elements α and β meeting all the members of $\mathcal{F}$. At least one of them (say α) belongs to D_1. Suppose first that $\alpha \in D_1 \cap D_2 \cap D_3$, say $\alpha \in A(j)$. If $\beta \in D_1 \cap D_2 \cap D_3$, too, then one of H_i is not covered. So, $\beta \notin D_1 \cap D_2 \cap D_3$. In order to cover B_j, it is necessary that $\beta \in B_j \setminus (D_1 \cap D_2 \cap D_3)$. Let $\beta \in A_{k,j}$. Then $\{\alpha, \beta\}$ does not meet $C_{k,j}$, a contradiction.

Thus, we may assume that $\alpha \in D_1 \setminus (D_2 \cap D_3)$ and that $\beta \notin D_1 \cap D_2 \cap D_3$. Suppose that $\alpha \in (D_1 \cap D_2) \setminus (D_1 \cap D_2 \cap D_3)$. Then $\beta \in D_3 \setminus (D_1 \cap D_2 \cap D_3)$. If say, $\alpha \in M(j, 1, 2)$ and $\beta \in M(j_2, 3)$, then $\{\alpha, \beta\}$ does not meet $L(j_1, j_2)$.

Finally, assume that $\alpha \in D_1 \setminus D_2$ and that $\beta \notin D_1 \cap D_2$. Then $\beta \in D_2 \setminus D_1$. If $\alpha \in A(j_1, 1)$ and $\beta \in A(j_2, 2)$, then $\{\alpha, \beta\}$ does not meet $B(j_1, j_2)$. This proves the lemma. ∎

It seems that the trick used in the proofs of Lemmas 4 and 5 does not work further, so we use a bit different twist.

Lemma 6. *Suppose that there exists $\mathcal{B}'' \subset \mathcal{B}$ with $\tau(\mathcal{B}') > 2f/3$ such that*

$$|A_1 \cap \ldots \cap A_8| \geq 0.5 \quad \text{for every} \quad A_1, \ldots, A_8 \in \mathcal{B}''. \tag{4}$$

Then (1) holds.

Proof of Lemma 6. Assume that such $\mathcal{B}'' \subset \mathcal{B}$ exists. Let D_1, D_2, D_3, $D_4 \in \mathcal{B}''$ be such that $|D_1 \cap \ldots \cap D_4| = \min\{|A_1 \cap \ldots \cap A_4| : A_1, A_2, A_3, A_4 \in \mathcal{B}''\}$.

Let $y = \frac{1}{r}|D_1 \cap \ldots \cap D_4|$. If $y \geq 3/4$, then by Lemma 5 we are done. Let

$$\frac{1}{2} \leq y \leq \frac{3}{4}.$$

Let $\{A_1, \ldots, A_s\}$ be a minimum family of subsets of size f covering all pairs of elements in $D_1 \cap \ldots \cap D_4$. By Lemma 2, $s = (yr/f)^2 + o((r/f)^2)$. Since $\tau(\mathcal{B}) > f$, for each $i = 1, \ldots, s$, there exists $H_i \in \mathcal{B}_{A_i}$.

Assume first that there exists a set M with $|M| \leq f/3$ such that for every $B_1, B_2, B_3, B_4 \in \mathcal{B}''_M$,

$$|(B_1 \cap \ldots \cap B_4) \setminus (D_1 \cap \ldots \cap D_4)| \geq r/4.$$

It follows then from (4) that for every $B_1, \ldots, B_4 \in \mathcal{B}''_M$,

$$|B_1 \cap \ldots \cap B_4| \geq r/2 + r/4 = 3r/4.$$

Since $\tau(\mathcal{B}''_M) > f/3$, Lemma 5 implies (1). Therefore, we may assume that for every M with $|M| \leq f/3$, there exist $B^1_M, \ldots, B^4_M \in \mathcal{B}''_M$, such that for the set $S_M = (B^1_M \cap \ldots \cap B^4_M) \setminus (D_1 \cap \ldots \cap D_4)$ we have $|S_M| < r/4$.

Partition D_1 into $1+3r/f$ parts $A(j)$, $j = 1, \ldots, 1+3r/f$ of size at most $f/3$ so that for $j = 1, \ldots, \lceil 3ry/f \rceil$, $A(j) \subset D_1 \cap \ldots \cap D_4$, and for $j = 1 + \lceil 3ry/f \rceil, \ldots, 1 + 3r/f$, $A(j) \subset D_1 \setminus (D_1 \cap \ldots \cap D_4)$. Partition every $S_{A(j)}$ into at most $\frac{3r}{8f}$ parts $A(j,i)$ of size at most $2f/3$. Since $\tau(\mathcal{B}) > f$, for every j and i, there exists $B(j,i) \in \mathcal{B}_{A(j) \cup A(j,i)}$. Define $\mathcal{F} = \{D_1, \ldots, D_4\} \cup \{H_i \mid i = 1, \ldots, s\} \cup \{B^k_{A(j)}, \mid k = 1, 2, 3, 4; j = 1, \ldots, 1 + 3r/f\} \cup \left\{B(j,i) \mid i = 1, \ldots, \frac{3r}{8f}; j = 1, \ldots, 1 + 3r/f\right\}$.

Then

$$|\mathcal{F}| \leq 4 + \left(\frac{ry}{f}\right)^2 + \frac{12r}{f} + \frac{9r^2}{8f^2} + o\left(\left(\frac{r}{f}\right)^2\right) = \left(\frac{9}{8} + y^2\right)\frac{r^2}{f^2} + o\left(\frac{r^2}{f^2}\right).$$

Since $y < 3/4$, $|\mathcal{F}| \leq \frac{27r^2}{16f^2} + o\left(\frac{r^2}{f^2}\right)$.

Suppose that there are two elements α and β meeting all the members of $\mathcal{F}$. At least one of them (say, α) belongs to D_1 (say, $\alpha \in A(j)$). Suppose first that $\alpha \in D_1 \cap \ldots \cap D_4$. If $\beta \in D_1 \cap \ldots \cap D_4$, too, then one of H_i is not covered. So, $\beta \notin D_1 \cap \ldots \cap D_4$. In order to cover $B^1_{A(j)}, \ldots, B^4_{A(j)}$, it is necessary that $\beta \in S_{A(j)}$. Let $\beta \in A(j,i)$. Then $\{\alpha, \beta\}$ does not meet $B(j,i)$, a contradiction.

Thus, we may assume that $\alpha \in D_1 \setminus (D_1 \cap \ldots \cap D_4)$ and that $\beta \notin D_1 \cap \ldots \cap D_4$ (otherwise we swap the roles of α and β). Then again β must be in $S_{A(j)}$ and the pair $\{\alpha, \beta\}$ does not meet some $B(j,i)$. This proves the lemma. ∎

4. Proof of Theorem 2

Assume first that for every set M with $|M| \le f/3$, there exist $B_1, \ldots, B_{12} \in \mathcal{B}_M$ such that,

$$|B_1 \cap \ldots \cap B_{12}| \le \frac{3r}{8}.$$

Fix an arbitrary $B_0 \in \mathcal{B}$ and divide it into $3r/f$ parts $B_0(j)$, $j = 1, \ldots, 3r/f$ of size $f/3$. By the assumption, for each $j = 1, \ldots, 3r/f$, we can choose

$$B^1_j, \ldots, B^{12}_j \in \mathcal{B}_{B_0(j)} \quad \text{such that} \quad |B^1_j \cap \ldots \cap B^{12}_j| \le \frac{3r}{8}. \tag{5}$$

Divide $B^1_j \cap \ldots \cap B^{12}_j$ into $\frac{9r}{16f}$ parts $A_{i,j}$, $i = 1, \ldots, \frac{9r}{16f}$ of size at most $2f/3$. Since, for each i, $|B_0(j) \cup A_{i,j}| \le f$, there exists $C_{i,j} \in \mathcal{B}_{B_0(j) \cup A_{i,j}}$.

Define

$$\mathcal{F} = \{B_0\} \cup \{B^k_j \mid k = 1, \ldots, 12;\ j = 1, \ldots, 3r/f\} \cup \left\{ C_{i,j} \mid i = 1, \ldots, \frac{9r}{16f};\ j = 1, \ldots, 3r/f \right\}.$$

Clearly, $|\mathcal{F}| \le 1 + \frac{36r}{f} + \frac{9 \cdot 3r^2}{16f^2} = \frac{27r^2}{16f^2} + o\left(\frac{r^2}{f^2}\right)$.

Suppose that there are two elements α and β meeting all the members of $\mathcal{F}$. Some of them (say α) belongs to B_0. By construction, there exists $B_0(j), 1 \le j \le 3r/f$, such that $\alpha \in B_0(j)$. In order to cover $B^1_j, \ldots, B^{12}_j$, it is necessary that $\beta \in B^1_j \cap \ldots \cap B^{12}_j$. Let $\beta \in A_{i,j}$. Then $\{\alpha, \beta\}$ does not meet $C_{i,j}$, a contradiction. Therefore, the assumption was wrong.

For every set M with $|M| \le f/3$, let $y(M) = \min\{|A_1 \cap \ldots \cap A_{12}| : A_1, \ldots, A_{12} \in \mathcal{B}_M\}$. Let M_0 with $|M_0| \le f/3$ be a set with $y = y(M_0) = \max\{y(M) \,|\, |M| \le f/3\}$. By above, $y \ge \frac{3r}{8}$.

Let $D_1, \ldots, D_8 \in \mathcal{B}_{M_0}$ be such that

$$z = |D_1 \cap \ldots \cap D_8| = \min\{|A_1 \cap \ldots \cap A_8| :\ A_1, \ldots, A_8 \in \mathcal{B}_{M_0}\}.$$

By Lemma 6, $y \le z \le r/2$.

CASE 1. There exists $M\subset D_1\setminus(D_1\cap\ldots\cap D_8)$ with $|M|\le f/3$ such that for every $B_1,\ldots,B_4\in\mathcal{B}_{M_0\cup M}$,

$$|(B_1\cap\ldots\cap B_4)\setminus(D_1\cap\ldots\cap D_8)|\ge\frac{3r}{4}-y.$$

Then by the definition of y, for every $B_1,\ldots,B_4\in\mathcal{B}_{M_0\cup M}$,

$$|B_1\cap\ldots\cap B_4|\ge y+\left(\frac{3r}{4}-y\right)=3r/4.$$

Thus by Lemma 5, we are done.

CASE 2. For every $M\subset D_1\setminus(D_1\cap\ldots\cap D_8)$ with $|M|\le f/3$, there exist $B^1_M,\ldots,B^4_M\in\mathcal{B}_{M_0\cup M}$, such that for the set $S_M=(B^1_M\cap\ldots\cap B^4_M)\setminus(D_1\cap\ldots\cap D_8)$ we have $|S_M|<\frac{3r}{4}-y$.

Partition D_1 into $1+3r/f$ parts $A(j)$, $j=1,\ldots,1+3r/f$ of size at most $f/3$ so that for $j=1,\ldots,\lceil 3z/f\rceil$, $A(j)\subset D_1\cap\ldots\cap D_8$, and for $j=1+\lceil 3z/f\rceil,\ldots,1+3r/f$, $A(j)\subset D_1\setminus(D_2\cap\ldots\cap D_8)$. For $j=1+\lceil 3rz/f\rceil,\ldots,1+3r/f$, partition every $S_{A(j)}$ into at most $\left\lceil\frac{3(0.75r-y)}{2f}\right\rceil$ parts $A(j,i)$ of size at most $2f/3$. Since $\tau(\mathcal{B})>f$, for every j and i, there exists $B(j,i)\in\mathcal{B}_{A(j)\cup A(j,i)}$. For every $j=1,\ldots,\lceil 3z/f\rceil$, by the definition of y, there exist $B^1_j,\ldots,B^{12}_j\in\mathcal{B}_M$, such that for the set $T_j=B^1_j\cap\ldots\cap B^{12}_j$, we have $|T_j|\le y$. Partition every T_j into at most $\left\lceil\frac{3y}{2f}\right\rceil$ parts $A(j,i)$ of size at most $2f/3$. Since $\tau(\mathcal{B})>f$, for every j and i, there exists $C(j,i)\in\mathcal{B}_{A(j)\cup A(j,i)}$.

Define $\mathcal{F}=\{D_1,\ldots,D_8\}\cup\{B^k_{A(j)},|\ k=1,2,3,4;\ j=1+\lceil 3z/f\rceil,\ldots,1+3r/f\}\cup\{B(j,i)\,|\,i=1,\ldots,\left\lceil\frac{3(0.75r-y)}{2f}\right\rceil;\ j=1+\lceil 3z/f\rceil,\ldots,1+3r/f\}\cup\{B^k_j,|\ k=1,\ldots,12;\ j=1,\ldots\lceil 3z/f\rceil\}\cup\left\{C(j,i)\,|\,i=1,\ldots,\left\lceil\frac{3y}{2f}\right\rceil;\ j=1,\ldots\lceil 3z/f\rceil\right\}$.

Then

$$|\mathcal{F}|\le 8+\left(\frac{12(r-z)}{f}+4\right)+\left(\frac{3(r-z)}{f}+1\right)\left\lceil\frac{3(0.75r-y)}{2f}\right\rceil+\left\lceil\frac{36z}{f}\right\rceil+\left\lceil\frac{3z}{f}\right\rceil\left\lceil\frac{3y}{2f}\right\rceil=\frac{9}{2}((r-z)(0.75r-y)+zy)\frac{1}{f^2}+o\left(\frac{r^2}{f^2}\right).$$

Since the function $g(y,z)=\frac{9}{2}((r-z)(0.75r-y)+zy)$ is linear in z and $y\le z\le r/2$, we have $g(y,z)\le\max\{g(y,y),g\left(y,\frac{r}{2}\right)\}$. Clearly,

$$g\left(y,\frac{r}{2}\right)=\frac{9}{2}\left(\frac{r}{2}(0.75r-y)+\frac{yr}{2}\right)=\frac{9}{2}\cdot\frac{3r^2}{8}=\frac{27r^2}{16}.$$

Consider $g(y,y) = \frac{9}{2}((r-y)(0.75r-y)+y^2)$. Since it is quadratic in y, it attains its maximum either at $y=\frac{3r}{8}$ or at $y=\frac{r}{2}$. But

$$g\left(\frac{3r}{8},\frac{3r}{8}\right) = \frac{9}{2}\left(\frac{5r}{8}\cdot\frac{3r}{8}+\frac{9r^2}{64}\right) = \frac{27r^2}{16}$$

and

$$g\left(\frac{r}{2},\frac{r}{2}\right) = \frac{9}{2}\left(\frac{r}{2}\cdot\frac{r}{4}+\frac{r^2}{4}\right) = \frac{27r^2}{16}.$$

Therefore, $|\mathcal{F}| \le \frac{27r^2}{16f^2} + o\left(\frac{r^2}{f^2}\right)$.

Suppose that there are two elements α and β meeting all the members of $\mathcal{F}$. At least one of them (say α) belongs to D_1. Then $\alpha \in A(j)$ for some j. If $1 \le j \le \lceil 3z/f \rceil$, then β must belong to $T_j = B_j^1 \cap \ldots \cap B_j^{12}$, say $\beta \in A(j,i)$. In this case $\{\alpha,\beta\}$ does not meet $C_{j,i}$, a contradiction. So, we may assume that $\lceil 3z/f \rceil + 1 \le j \le 3r/f$ (i.e., $\alpha \in D_1 \setminus (D_1 \cap \ldots \cap D_8)$) and that $\beta \notin D_1 \cap \ldots \cap D_8$. Then β must be in $S_{A(j)}$ and the pair $\{\alpha,\beta\}$ does not meet some $B(j,i)$. This proves Theorem 2. ∎

Remark. One can slightly improve the factor $\frac{27}{16}$ in (1) along the lines of the proofs. But it would make the proofs more complicated and it seems that to make the factor $1+o(1)$ one needs an additional idea.

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